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The isotherms migration method in the theory and practice of heat and mass transfer investigation—II. Numerical–analytical determination of temperature fields

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Abstract—The equation of the process of non-stationary heat conduction in solid bodies—a new equation of mathematical physics—is found for a multidimensional case of the migration of isothermal surfaces. The analysis of the existence and of the uniqueness of the solution of the first boundary value problem for the above equation is given. The application of the method of perturbations in the kinematic description of heat conduction to the solution of problems with temperature-dependent thermophysical properties of the medium is shown. The finite difference approximation of the new heat conduction equation by the boundary value and explicit schemes is found and the stability condition of the latter is substantiated. Using these schemes an algorithm is developed to determine the temperature fields and location of the interface between new and old phases by a computer.

1. INTRODUCTION

THE FIRST part of the present series [1] gives a number of regularities of the migration of isothermal surfaces that are applied in practice for determining temperature fields with temperature-dependent properties of material and boundary conditions. In this case use was made of the description of non-stationary heat conduction as spatial–temporal temperature variation $T = f(M, \tau)$. In the presence of phase transition, the thermal state of phases is determined in the form of the dependence $T = f(M, \tau)$ and the Stefan condition is formulated in the migrations of the isothermal surface separating them. Naturally, there occurs an idea on the possibility of treating and solving the problems of non-stationary heat conduction with and without phase transition in the migration of isotherms. The advantages of such an approach to the problem of non-linear heat transfer are shown in what follows.

2. PROBLEM FORMULATION

Consider a quasi-linear equation of non-stationary heat conduction

$$C(T)\frac{\partial T}{\partial \tau} = \operatorname{div}\,[\lambda(T)\operatorname{grad} T] \qquad (1)$$

which will be rewritten in the form

$$C(T)\frac{\partial T}{\partial \tau} = \lambda(T)\nabla^2 T + \lambda'(T)\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right]. \qquad (1')$$

Its solution $T = \Phi(x, y, z, \tau)$ may be presented implicitly as $F(x, y, z, \tau, T) = 0$, where

$$F(x, y, z, \tau, T) \equiv \Phi(x, y, z, \tau) - T. \qquad (2)$$

Assume temperature T, along with time τ and coordinates y, z, to be independent variables, and coordinate x to be an unknown function

$$x = x(y, z, \tau, T) \qquad (3)$$

monotonously dependent on T.

Calculate the derivatives $\partial x/\partial \tau$, $\partial x/\partial y$, $\partial x/\partial z$ and $\partial x/\partial T$ using the rule of the differentiation of the implicitly assigned function, then we obtain on the basis of equation (2):

$$\frac{\partial x}{\partial \tau} = -\frac{\partial F}{\partial \tau}\bigg/\frac{\partial F}{\partial x} = -\frac{\partial \Phi}{\partial \tau}\bigg/\frac{\partial \Phi}{\partial x} = -\frac{\partial T}{\partial \tau}\bigg/\frac{\partial T}{\partial x}$$

$$\frac{\partial x}{\partial y} = -\frac{\partial F}{\partial y}\bigg/\frac{\partial F}{\partial x} = -\frac{\partial \Phi}{\partial y}\bigg/\frac{\partial \Phi}{\partial x} = -\frac{\partial T}{\partial y}\bigg/\frac{\partial T}{\partial x}$$

$$\frac{\partial x}{\partial z} = -\frac{\partial F}{\partial z}\bigg/\frac{\partial F}{\partial x} = -\frac{\partial \Phi}{\partial z}\bigg/\frac{\partial \Phi}{\partial x} = -\frac{\partial T}{\partial z}\bigg/\frac{\partial T}{\partial x}$$

$$\frac{\partial x}{\partial T} = 1\bigg/\frac{\partial T}{\partial x}. \qquad (4)$$

Here it is taken into account that $\partial F/\partial \tau = \partial \Phi/\partial \tau$, etc., since T is now an independent variable and $x = x(y, z, \tau, T)$.

Using set (4) as a basis one may write sequentially

$$\frac{\partial F}{\partial \tau} = -\frac{x'_\tau}{x'_T},\quad \frac{\partial F}{\partial y} = -\frac{x'_y}{x'_T},\quad \frac{\partial F}{\partial z} = -\frac{x'_z}{x'_T},\quad \frac{\partial F}{\partial x} = \frac{1}{x'_T}. \qquad (5)$$

NOMENCLATURE

$C(T)$	volumetric heat capacity [$J\,m^{-3}\,K^{-1}$]	T_f	surrounding temperature [K]
l_0	half-thickness of a plate, cylinder (sphere) radius [m]	T_0	initial temperature [K]
L	volumetric heat of phase transition [$J\,m^{-3}$]	T_s	solidification (melting) temperature [K]
m	coefficient of the body shape equal to one, two or three for a plate, cylinder or sphere, respectively	x, y, z	Cartesian coordinates [m].
q	heat flux density [$W\,m^{-2}$]		
$T(x, y, z, \tau)(T(M, \tau))$	current temperature [K]		

Greek symbols

α	heat transfer coefficient [$W\,m^{-2}\,K^{-1}$]
$\lambda(T)$	thermal conductivity [$W\,m^{-1}\,K^{-1}$]
τ	time [s].

Then, using these equations and calculating the derivatives of $F'_y(x, y, z, \tau, T)$ and $F'_z(x, y, z, \tau, T)$ with respect to y and z, respectively, we obtain

$$\frac{d}{dy}F'_y = F''_{yy}+F''_{yx}x'_y = -\frac{\partial}{\partial y}\left(\frac{x'_y}{x'_T}\right)$$

$$\frac{d}{dz}F'_z = F''_{zz}+F''_{zx}x'_z = -\frac{\partial}{\partial z}\left(\frac{x'_z}{x'_T}\right). \quad (6)$$

Calculating also the derivatives of the complex function $F'_x(x, y, z, \tau, T)$ with respect to y, x and T, we have

$$\frac{d}{dy}F'_x = \frac{\partial}{\partial y}\left(\frac{1}{x'_T}\right), \quad \frac{d}{dz}F'_x = \frac{\partial}{\partial z}\left(\frac{1}{x'_T}\right),$$

$$\frac{d}{dT}F'_x = \frac{\partial}{\partial T}\left(\frac{1}{x'_T}\right), \quad F''_{xT} \equiv 0. \quad (7)$$

It follows from equation (7) that

$$F''_{xy} = \frac{\partial}{\partial y}\left(\frac{1}{x'_T}\right) - \frac{1}{x'_T}\frac{\partial}{\partial T}\left(\frac{1}{x'_T}\right)x'_y \quad (8)$$

$$F''_{xz} = \frac{\partial}{\partial z}\left(\frac{1}{x'_T}\right) - \frac{1}{x'_T}\frac{\partial}{\partial T}\left(\frac{1}{x'_T}\right)x'_T \quad (9)$$

$$F''_{xx} = \frac{1}{x'_T}\frac{\partial}{\partial T}\left(\frac{1}{x'_T}\right). \quad (10)$$

Having substituted the values of F''_{xy} and F''_{xz} into equation (6), we obtain

$$F''_{yy} = -\frac{\partial}{\partial y}\left(\frac{x'_y}{x'_T}\right) - x'_y\left[\frac{\partial}{\partial y}\left(\frac{1}{x'_T}\right) - \frac{x'_y}{x'_T}\frac{\partial}{\partial T}\left(\frac{1}{x'_T}\right)\right] \quad (11)$$

$$F''_{zz} = -\frac{\partial}{\partial z}\left(\frac{x'_z}{x'_T}\right) - x'_z\left[\frac{\partial}{\partial z}\left(\frac{1}{x'_T}\right) - \frac{x'_z}{x'_T}\frac{\partial}{\partial T}\left(\frac{1}{x'_T}\right)\right]. \quad (12)$$

Replacing the derivatives $\partial^2T/\partial x^2 \equiv \partial^2\Phi/\partial x^2$, $\partial^2T/\partial y^2 \equiv \partial^2\Phi/\partial y^2$, $\partial^2T/\partial z^2 \equiv \partial^2\Phi/\partial z^2$, $\partial T/\partial\tau \equiv \partial\Phi/\partial\tau$, $\partial T/\partial x \equiv \partial\Phi/\partial x$, $\partial T/\partial y \equiv \partial\Phi/\partial y$, $\partial T/\partial z \equiv \partial\Phi/\partial z$ in equation (1′) by the expressions (5), (10), (11) and (12), we have as a result of all transformations an equation with respect to the velocity x'_τ of the migration of an isothermal surface toward x:

$$C(T)x'_\tau = \lambda(T)\left\{x''_{yy}+x''_{zz}+\frac{x''_{TT}}{(x'_T)^2}[1+(x'_y)^2+(x'_z)^2] - \frac{2}{x'_T}(x'_yx''_{Ty}+x'_zx''_{Tz})\right\} - \frac{\lambda'(T)}{x'_T}[1+(x'_y)^2+(x'_z)^2]. \quad (13)$$

Equations with respect to the velocity of the migration of an isothermal surface toward y and z are obtained from equation (13) by the substitution of x into y (and y into x) and x into z (and z into x), respectively, and have the form:

$$C(T)y'_\tau = \lambda(T)\left\{y''_{xx}+y''_{zz}+\frac{y''_{TT}}{(y'_T)^2}[1+(y'_x)^2+(y'_z)^2] - \frac{2}{y'_T}(y'_xy''_{Tx}+y'_zy''_{Tz})\right\} - \frac{\lambda'(T)}{y'_T}[1+(y'_x)^2+(y'_z)^2] \quad (14)$$

$$C(T)z'_\tau = \lambda(T)\left\{z''_{yy}+z''_{xx}+\frac{z''_{TT}}{(z'_T)^2}[1+(z'_y)^2+(z'_x)^2] - \frac{2}{z'_T}(z'_yz''_{Ty}+z'_xz''_{Tx})\right\} - \frac{\lambda'(T)}{z'_T}[1+(z'_y)^2+(z'_x)^2]. \quad (15)$$

Operating analogously to the above one may show that for the practically important case of one-dimensional distribution of heat in a plate ($m = 1$), cylinder ($m = 2$) and sphere ($m = 3$) the equation for the time-variation of the isothermal surface location calculated from the symmetry centre (axis, plane) is

$$C(T)\frac{\partial x}{\partial \tau} = \lambda(T)\frac{\partial^2 x}{\partial T^2}\left(\frac{\partial x}{\partial T}\right)^{-2} - \lambda'(T)\left(\frac{\partial x}{\partial T}\right)^{-1} - \frac{(m-1)\lambda(T)}{x}. \quad (16)$$

Note that equation (16) was first obtained in 1968 [2].

Analysis of relations (3) and (13)–(15) allows the determination of the effect of geometric properties of isothermal surfaces on the formation of temperature fields.

Thus, for instance, equation (3) in the form

$$x = x(y, z, \tau, T)$$

or

$$F(x, y, z, \tau, T) = x(y, z, \tau, T) - x = 0$$

prescribes a two-parametric family of surfaces $S_{\tau,T}$, the length of the normal vector to which is equal to

$$|\nabla F|^2 = 1 + (x'_y)^2 + (x'_z)^2$$

(-1, x'_y and x'_z are the components of this vector). The presence of the co-factor $|\nabla F|^2$ and the functions x'_y, x'_z, x''_{yy} and x''_{zz} on the right-hand side of equation (13) indicates that the projection of the velocity vector of the isotherm migration toward x is affected not only by the temperature gradient $(x'_T)^{-1}$ and its variations over T, y and z, but also by the curvature of an isothermal surface in its sections by the planes normal to the axes $0y$ and $0z$.

It is expedient to note also that the value $[1+(x'_y)^2+(x'_z)^2]^{1/2}$ relates the element of the area $\Delta\sigma$ of an isothermal surface with the area of its projection $\Delta\sigma_1$ onto the plane y, z according to the formula

$$\Delta\sigma = \Delta\sigma_1[1+(x'_y)^2+(x'_z)^2]^{1/2}.$$

3. ANALYSIS OF THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

The analysis of the existence and uniqueness of the solution will be conducted as applied to equation (16) for the particular case of a plate ($m = 1$) with constant thermophysical properties $C(T) = 1$, $\lambda(T) = 1$. Here the first boundary value problem is written as

$$x'_\tau = x''_{TT}(x'_T)^{-2}, \quad \tau > 0, \quad d < T < e \quad (17)$$

$$x(T, 0) = \varphi(T) \quad (18)$$

$$x(T = f_1(\tau), \tau) = 0, \quad \tau > 0 \quad (19)$$

$$x(T = f_2(\tau), \tau) = b, \quad \tau > 0. \quad (20)$$

Assume that $d \leqslant f_1(\tau) < T < f_2(\tau) \leqslant e$ represents a curved band $\tau > 0$ and $f_1(\tau) \leqslant T \leqslant f_2(\tau)$ a rectangular band $\tau > 0$, $0 \leqslant \xi \leqslant 1$, where $\xi = [T-f_1(\tau)]/[f_2(\tau)-f_1(\tau)]$. It is mutually unambiguous since $f_1(\tau)-f_2(\tau) \neq 0$. In this case the curves $T = f_1(\tau)$ and $T = f_2(\tau)$ respectively change to straight

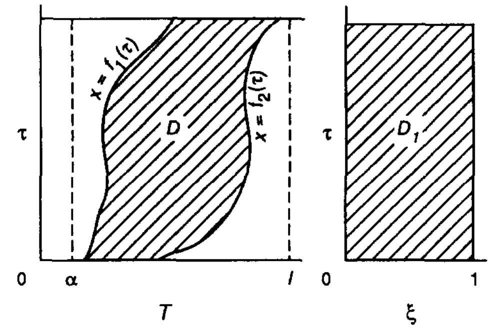

FIG. 1. Reflection of the curved band $\tau > 0, f_1(\tau) \leqslant T < f_2(\tau)$ to the rectangular band $\tau > 0$, $0 \leqslant \xi \leqslant 1$.

lines $\xi = 0$ and $\xi = 1$ (Fig. 1). Then instead of (17)–(20) we obtain a new problem:

$$x'_\tau = x''_{\xi\xi}(x'_\xi)^{-2} + F(\xi,\tau)(x'_\xi)^{-1}, \quad \tau > 0, \quad 0 < \xi < 1 \quad (21)$$

$$x(\xi, 0) = \varphi_1(\xi) \quad (22)$$

$$x(\xi = 0, \tau) = 0, \quad \tau > 0 \quad (23)$$

$$x(\xi = 1, \tau) = b, \quad \tau > 0, \quad (24)$$

where

$$F(\xi,\tau) = \{f'_1 f_2 - f_1 f'_2 + [\xi(f_2-f_1)+f_1](f'_2-f'_1)\}(f_2-f_1)^{-2}.$$

Equation (21) will be analysed for $f_1 = $ const. and $f_2 = $ const. written in the form

$$x'_\tau = x''_{TT}(x'_T)^{-2} = \frac{\partial}{\partial T}\left(-\frac{1}{x'_T}\right). \quad (25)$$

Equation (25), with a diverging main part, is not uniformly parabolic and falls out of the class of equations considered in refs. [3–6]. It may be stated with high probability that this equation has no a priori estimates ($|x|$, $|x'_T|$) and the problem on the existence of the solution of the first boundary value problem (17)–(20) involves the problem of the selection of initial and boundary conditions.

Proof of the uniqueness of the solution of problem (21)–(24) (or, similarly, problem (17)–(20)) in the class of the functions will be determined below.

Assume that problem (21)–(24) has two solutions x_1 and x_2. Then for the difference between these solutions, $w = x_1 - x_2$, equation (21) becomes linear:

$$w'_\tau = \frac{w''_{\xi\xi}}{(x'_{1\xi})^2} - \frac{x'_{2\xi\xi}(x'_{1\xi}-x'_{2\xi})w'_\xi}{(x'_{1\xi})^2(x'_{2\xi})^2} - F(\xi,\tau)\frac{w}{x_1 x_2} \quad (26)$$

and the initial and boundary values of w vanish:

$$w(\xi, 0) = 0 \quad (27)$$

$$w(\xi = 0, \tau) = 0, \quad \tau > 0 \quad (28)$$

$$w(\xi = 1, \tau) = 0, \quad \tau > 0. \quad (29)$$

Designate the set of functions $\psi(\xi, \tau)$ which were determined within the range $\tau > 0$, $0 \leqslant \xi \leqslant 1$ and which have continuous partial derivatives ψ'_τ, ψ'_ξ, $\psi''_{\xi\xi}$ (with $(\psi'_\xi)^2 > 0$ within the band $\tau > 0$, $0 \leqslant \xi \leqslant 1$) as M. Then the following confirmation will be valid: if functions $f_1(\tau)$ and $f_2(\tau)$ are constantly differentiated for $\tau > 0$ and $[f_1(\tau) - f_2(\tau)]^2 > 0$, then the solution of problem (21)–(24) in the class M is unique.

The proof of this confirmation follows from the fact that, based on the principle of maximum, the boundary value problem with respect to w has a zero solution only [7]; that is, properly, the essence of the theorem of the uniqueness of the solution of the first boundary value problem for a linear parabolic equation.

'Improving' equation (17) to the form

$$x'_\tau = [1+\varepsilon_1(x'_T)^2]x''_{TT}/[\varepsilon_2+(x'_T)^2], \qquad (30)$$

where ε_1 and ε_2 are positive constants, the uniqueness and existence of the solution of problem (30), (18)–(20), may be proved.

4. APPLICATION OF THE PERTURBATION METHOD

In ref. [8] the application of group analysis in obtaining both invariant solutions of the heat conduction equation with the migration of isothermal surfaces and invariant solutions of boundary value problems of this equation is considered in detail. It was found that construction of these solutions is especially facilitated by a new treatment of heat transfer.

The description of the process of non-stationary heat conduction with the migration of isothermal surfaces allows one to apply the perturbation method to the solution of problems with temperature-dependent thermophysical properties.

For instance, for a plate ($m = 1$), hollow cylinder ($m = 2$) and hollow sphere ($m = 3$) (one-dimensional case) the problem of heat conduction with boundary conditions of the first kind in dimensionless form is

$$\tilde{C}(T)x(x'_T)^2x'_\tau = \tilde{\lambda}(T)xx''_{TT} - \tilde{\lambda}'(T)xx'_T$$
$$-(m-1)\tilde{\lambda}(T)(x'_T)^2, \quad \tau > 0, \quad a < T < b \qquad (31)$$

$$x(T,0) = f(T) \qquad (32)$$

$$x(T = a, \tau) = 1, \quad \tau > 0 \qquad (33)$$

$$x(T = b, \tau) = 1+\Delta, \quad \tau > 0, \qquad (34)$$

where $x(T, \tau)$ is the location of isothermal surfaces on which the temperature $T = idem$ is prescribed, τ is the dimensionless time, $\tilde{C}(T)$ and $\tilde{\lambda}(T)$ are the temperature dependences of the relative values of volumetric heat capacity and thermal conductivity.

It is easy to see that problem (31)–(34) corresponds to non-stationary heat conduction in the bodies mentioned when the initial temperature distribution T along the coordinate x is assigned and on the bounding surfaces with the coordinates $x = 1$ and $x = 1+\Delta$ constant values of temperature a and b, respectively, are maintained.

Assume that the functions $\tilde{C}(T)$ and $\tilde{\lambda}(T)$ have the form

$$\tilde{C}(T) = 1+\varepsilon\tilde{C}_1(T), \quad \tilde{\lambda}(T) = 1+\delta\tilde{\lambda}_1(T), \qquad (35)$$

where $\tilde{C}_1(T)$ and $\tilde{\lambda}_1(T)$ are limited within the section (a, b), and ε and δ are rather small.

We seek the solution of problem (31)–(34) in the form

$$x(T,\tau) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij}(T,\tau)\varepsilon^i\delta^j. \qquad (36)$$

Substitution of the functions $\tilde{C}(T)$ and $\tilde{\lambda}(T)$ and of the series (36) into equation (31) yields

$$(1+\varepsilon\tilde{C}_1(T))\left(\sum_{i,j=0}^{\infty} x_{ij}\varepsilon^i\delta^j\right)\left(\sum_{i,j=0}^{\infty} x'_{ij}\varepsilon^i\delta^j\right)^2\left(\sum_{i,j=0}^{\infty} \dot{x}_{ij}\varepsilon^i\delta^j\right)$$
$$= (1+\delta\tilde{\lambda}_1)\left(\sum_{i,j=0}^{\infty} x_{ij}\varepsilon^i\delta^j\right)\left(\sum_{i,j=0}^{\infty} x''_{ij}\varepsilon^i\delta^j\right)$$
$$-\delta\tilde{\lambda}'_1\left(\sum_{i,j=0}^{\infty} x_{ij}\varepsilon^i\delta^j\right)\left(\sum_{i,j=0}^{\infty} x'_{ij}\varepsilon^i\delta^j\right)$$
$$-(m-1)(1+\delta\tilde{\lambda}_1)\left(\sum_{i,j=0}^{\infty} x'_{ij}\varepsilon^i\delta^j\right)^2. \qquad (37)$$

Here $\Sigma_{i,j=0}^{\infty}$ should be understood as $\Sigma_{i=0}^{\infty}\,\Sigma_{j=0}^{\infty}$, $\dot{x}_{ij} = \partial x_{ij}/\partial\tau$, $x'_{ij} = \partial x_{ij}/\partial T$, $x''_{ij} = \partial^2 x_{ij}/\partial T^2$ and $\tilde{\lambda}'_1 = \partial\tilde{\lambda}_1/\partial T$.

Then, equating the terms with equal powers $\varepsilon^i\delta^i$ we obtain an infinite system of equations for determining the unknown functions $x_{ij}(T, \tau)$. For series (36) to satisfy problem (31)–(34) the following conditions must be valid:

$$x(T,0) = \sum_{i,j=0}^{\infty} x_{ij}(T,0)\varepsilon^i\delta^j = f(T) \qquad (38)$$

$$x(a,\tau) = \sum_{i,j=0}^{\infty} x_{ij}(a,\tau)\varepsilon^i\delta^j = 1, \quad \tau > 0 \qquad (39)$$

$$x(b,\tau) = \sum_{i,j=0}^{\infty} x_{ij}(b,\tau)\varepsilon^i\delta^j = 1+\Delta, \quad \tau > 0. \qquad (40)$$

Assume that these conditions are fulfilled at

$$x_{00}(T,0) = f(T), \quad x_{ij}(T,0) = 0 \quad \text{for} \quad i+j \geqslant 1$$
$$x_{00}(a,\tau) = 1, \quad x_{ij}(a,\tau) = 0 \quad \text{for} \quad i+j \geqslant 1$$
$$x_{00}(b,\tau) = 1+\Delta, \quad x_{ij}(b,\tau) = 0 \quad \text{for} \quad i+j \geqslant 1.$$

We determine the functions x_{00}, x_{01} and x_{10}. For x_{00} there is the boundary value problem with the nonlinear equation of the process:

$$x_{00}(x'_{00})^2\dot{x}_{00} = x_{00}x''_{00} - (m-1)(x'_{00})^2 \qquad (41)$$

$$x_{00} = f(T), \quad \tau = 0 \qquad (42)$$

$$x_{00}(a,\tau) = 1, \quad \tau > 0 \qquad (43)$$

$$x_{00}(b,\tau) = 1+\Delta, \quad \tau > 0, \tag{44}$$

coinciding with problem (31)–(34) at $\tilde{C}(T) = 1$, $\tilde{\lambda}(T) = 1$ in the ordinary formulation when temperature T is a dependent variable and coordinate x and time τ are independent variables. The corresponding boundary value problem is linear and its analytical solution is derived in refs. [9, 10], with the use of which the unknown quantity $x_{00}(T,\tau)$ is obtained.

The boundary value problem for determining x_{01} becomes linear:

$$x_{00}(x'_{00})^2\dot{x}_{01}+\dot{x}_{00}[x_{01}(x'_{00})^2+2x_{00}x'_{00}x'_{01}]$$
$$= x_{00}x''_{01}+x_{01}x''_{00}+\tilde{\lambda}_1 x_{00}x''_{00}-\tilde{\lambda}'_1 x_{00}x'_{00}$$
$$-(m-1)[2x'_{00}x'_{01}+\tilde{\lambda}_1(x'_{00})^2] \tag{45}$$

$$x_{01}(T,0) = 0 \tag{46}$$

$$x_{01}(a,\tau) = 0, \quad \tau > 0 \tag{47}$$

$$x_{01}(b,\tau) = 0, \quad \tau > 0. \tag{48}$$

The boundary value problem with respect to x_{10} is also linear:

$$x_{00}(x'_{00})^2\dot{x}_{10}+\dot{x}_{00}[x_{10}(x'_{00})^2+\tilde{C}_1(x'_{00})^2x_{00}$$
$$+2x_{00}x'_{00}x'_{10}] = x_{00}x''_{10}-2(m-1)x'_{00}x'_{10}+x_{10}x''_{00} \tag{49}$$

$$x_{10}(T,0) = 0 \tag{50}$$

$$x_{10}(a,\tau) = 0, \quad \tau > 0 \tag{51}$$

$$x_{10}(b,\tau) = 0, \quad \tau > 0. \tag{52}$$

The question of the convergence of the functional series (36) is unsolved because its solution requires the knowledge of exact estimates of the values of $|x_{ij}(T,\tau)|$, thus making it a complex problem.

If it is assumed that there exist positive constants A, B and M so that

$$\max|x_{ij}(T,\tau)| \leqslant A^iB^jM, \quad i,j = 1,\ldots,$$
$$a \leqslant T \leqslant b, \quad 0 < \tau < t, \tag{53}$$

then it can be easily proved that at rather small ε and δ, series (36) converges and with accuracy up to an infinitely small quantity having order higher than $\varepsilon^2+\varepsilon\delta+\delta^2$, so that the solution of problem (31)–(34) may be presented in the form

$$x(T,\tau) \cong x_{00}+\varepsilon x_{10}+\delta x_{01}. \tag{54}$$

In fact, equations (36) and (53) yield

$$\left|\sum_{i,j=0}^{\infty} x_{ij}\varepsilon^i\delta^j\right| \leqslant \sum_{i,j=0}^{\infty}|x_{ij}|\times|\varepsilon|^i|\delta|^j$$
$$\leqslant M\sum_{i,j=0}^{\infty}|(B\varepsilon)|^i|(A\delta)|^j. \tag{55}$$

Series $\Sigma_{i,j=0}^{\infty}|(B\varepsilon)|^i|(A\delta)|^j$ converges absolutely when $|B\varepsilon| < 1$, $|A\delta| < 1$. Then, from the comparison criterion and inequality (55) the proof of the advanced assumption is obtained.

5. APPLICATION OF THE METHOD OF FINITE DIFFERENCES

The difficulties of constructing effective methods for computer solution of the problem with substance phase transition when the position of the interface between old and new phases is found from the Stefan conditions are well known. Moreover, when solving the problems without phase transition by computer a substantial portion of time is spent on selecting thermophysical characteristics of the body material from the input arrays on each time layer. The latter confirmation is based on the results of control computer calculations, made by the author, of the problem of a non-stationary temperature field in an unbounded plate (one-dimensional case) in which the space step was taken to be equal to 1/20 of the plate half-thickness and the time step amounted to $\Delta Fo = 0.005$. The initial temperature was assumed to be equal to zero and the temperature of the boundary during the entire process was taken to be equal to unity. The arrays of $\lambda = \lambda(T)$ and $C = C(T)$ for values of T (with each value of T, $\lambda = 1$, $C = 1$) were input to the memory of an ES-1050-type computer. Then the system of 20 algebraic equations with respect to unknown temperatures at 20 body points on each time layer was solved which corresponds to the finite difference approximation by the implicit absolutely stable non-iterative Laasonen scheme. In this case, in the first version of the program the selection of thermophysical properties from the input arrays with linear interpolation between the node values of λ and C was envisaged, while in the second version this procedure was not foreseen because the values of λ and C were taken to be equal to unity in the corresponding equations of the above mentioned system. The calculations performed showed that the first case requires approximately 20% more computer time than the second.

With the use of iterations when verifying thermophysical properties at each time step the computer time spent increases in direct proportion to the number of iterations.

It is necessary to note the fact that traditional methods of solving heat conduction problems yield a good deal of excess information in those cases when it is necessary only to follow the behaviour of some isothermal surfaces.

Expensive computer time expenditure and accumulation of excess information when solving the problems without substance phase transition may be avoided and, moreover, effective algorithms for solving problems by computer with substance phase transition may be constructed by using the description of the process with the migration of isothermal surfaces.

5.1. Construction of finite difference approximation for the heat conduction equation with the migration of isothermal surfaces

The difference scheme for computer solution of the above problems in a practically important case of one-dimensional heat propagation over a plate ($m = 1$), cylinder ($m = 2$) and sphere ($m = 3$) when the equation of the isothermal surface position variation $T = idem$ with time τ calculated from the centre of symmetry is

$$C(T)\frac{\partial x}{\partial \tau} = \frac{\partial}{\partial T}\left[-\frac{\lambda(T)}{\partial x/\partial T}\right] - \frac{(m-1)\lambda(T)}{x}. \quad (56)$$

Furthermore, the coordinate x will be calculated from the surface bounding the body (Fig. 2). Then with the characteristic dimension l_0 (plate thickness, outer radius of solid or hollow cylinder, sphere), equation (56) takes the form

$$C(T)\frac{\partial x}{\partial \tau} = \frac{\partial}{\partial T}\left[-\frac{\lambda(T)}{\partial x/\partial T}\right] + \frac{(m-1)\lambda(T)}{l_0 - x}. \quad (56')$$

Following ref. [11], we integrate equation (56′) over the variable T within the limits from $T_{i-1/2} = 0.5(T_i + T_{i-1})$ to $T_{i+1/2} = 0.5(T_i + T_{i+1})$ and over τ from $\tau^{(n)}$ to $\tau^{(n+1)}$, where i is the spatial layer number and n is the temporal layer number:

$$\int_{T_{i-1/2}}^{T_{i+1/2}}\left[\int_{\tau^{(n)}}^{\tau^{(n+1)}} C(T)\frac{\partial x}{\partial \tau}\,d\tau\right]dT = \int_{\tau^{(n)}}^{\tau^{(n+1)}}\left\{\int_{T_{i-1/2}}^{T_{i+1/2}}\frac{\partial}{\partial T}\left[-\frac{\lambda(T)}{\partial x/\partial T}\right]dT\right\}d\tau + (m-1)\int_{T_{i-1/2}}^{T_{i+1/2}}\left[\int_{\tau^{(n)}}^{\tau^{(n+1)}}\frac{\lambda(T)}{l_0 - x}\,d\tau\right]dT. \quad (57)$$

Then, we have, successively, for the left-hand side of equation (57) within the limits of a temporal layer with the duration of $\Delta\tau = \tau^{(n+1)} - \tau^{(n)}$:

$$\int_{\tau^{(n)}}^{\tau^{(n+1)}} C(T)\frac{\partial x}{\partial \tau}\,d\tau = C(T)\int_{\tau^{(n)}}^{\tau^{(n+1)}}\frac{\partial x}{\partial \tau}\,d\tau = C(T)[x(T,\tau^{(n+1)}) - x(T,\tau^{(n)})];$$

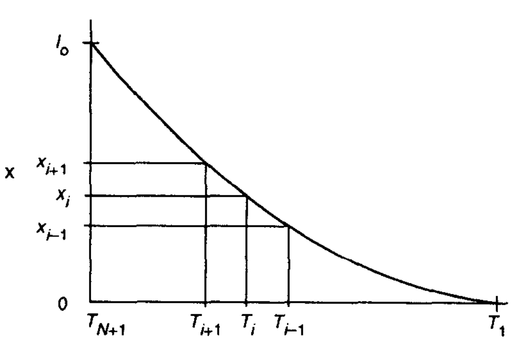

FIG. 2. Plot of x as a function of T.

$$\int_{T_{i-1/2}}^{T_{i+1/2}}\left[\int_{\tau^{(n)}}^{\tau^{(n+1)}} C(T)\frac{\partial x}{\partial \tau}\,d\tau\right] = \int_{T_{i-1/2}}^{T_{i+1/2}} C(T)[x(T,\tau^{(n+1)}) - x(T,\tau^{(n)})]\,dT \approx [x(T_i,\tau^{(n+1)}) - x(T_i,\tau^{(n)})](T_{i+1/2} - T_{i-1/2})C_i = 0.5C_i(x_i^{(n+1)} - x_i^{(n)})(T_{i+1} - T_{i-1}),$$

where

$$C_i = \int_{T_{i-1/2}}^{T_{i+1/2}} C(T)\,dT/(T_{i+1/2} - T_{i-1/2}).$$

If the last integral is not taken and an approximation is made following the rule of rectangles, we have

$$C_i \approx C(T_i).$$

Now consider the first integral from the right-hand side of equation (57):

$$\int_{T_{i-1/2}}^{T_{i+1/2}}\frac{\partial}{\partial T}\left[-\frac{\lambda(T)}{\partial x/\partial T}\right]dT = \left[\frac{\lambda(T)}{\partial x/\partial T}\right]_{T_{i-1/2}} - \left[\frac{\lambda(T)}{\partial x/\partial T}\right]_{T_{i+1/2}}$$

$$\left[\frac{\lambda(T)}{\partial x/\partial T}\right]_{T_{i-1/2}} \approx \frac{\lambda(T_{i-1/2})(T_i - T_{i-1})}{x(\tau,T_i) - x(\tau,T_{i-1})} = \frac{\lambda_{i-1/2}(T_i - T_{i-1})}{x(\tau,T_i) - x(\tau,T_{i-1})}$$

$$\left[\frac{\lambda(T)}{\partial x/\partial T}\right]_{T_{i+1/2}} \approx \frac{\lambda(T_{i+1/2})(T_{i+1} - T_i)}{x(\tau,T_{i+1}) - x(\tau,T_i)} = \frac{\lambda_{i+1/2}(T_{i+1} - T_i)}{x(\tau,T_{i+1}) - x(\tau,T_i)}.$$

This yields, successively:

$$\int_{\tau^{(n)}}^{\tau^{(n+1)}}\left\{\int_{T_{i-1/2}}^{T_{i+1/2}}\frac{\partial}{\partial T}\left[-\frac{\lambda(T)}{\partial x/\partial T}\right]dT\right\}d\tau \approx \int_{\tau^{(n)}}^{\tau^{(n+1)}}\frac{\lambda_{i-1/2}(T_i - T_{i-1})}{x(\tau,T_i) - x(\tau,T_{i-1})}\,d\tau \int_{\tau^{(n)}}^{\tau^{(n+1)}}\frac{\lambda_{i+1/2}(T_{i+1} - T_i)}{x(\tau,T_{i+1}) - x(\tau,T_i)}\,d\tau,$$

$$\int_{\tau^{(n)}}^{\tau^{(n+1)}}\frac{\lambda_{i-1/2}(T_i - T_{i-1})}{x(\tau,T_i) - x(\tau,T_{i-1})}\,d\tau = \lambda_{i-1/2}(T_i - T_{i-1})\int_{\tau^{(n)}}^{\tau^{(n+1)}}\frac{d\tau}{x(\tau,T_i) - x(\tau,T_{i-1})} = \frac{\lambda_{i-1/2}(T_i - T_{i-1})(\tau^{(n+1)} - \tau^{(n)})}{x[\tau^{(n)} + \theta(\tau^{(n+1)} - \tau^{(n)}), T_i] - x[\tau^{(n)} + \theta(\tau^{(n+1)} - \tau^{(n)}), T_{i-1}]},$$

$$\int_{\tau^{(n)}}^{\tau^{(n+1)}}\frac{\lambda_{i+1/2}(T_{i+1} - T_i)}{x(\tau,T_{i+1}) - x(\tau,T_i)}\,d\tau \approx \frac{\lambda_{i+1/2}(T_{i+1} - T_i)(\tau^{(n+1)} - \tau^{(n)})}{x[\tau^{(n)} + \theta(\tau^{(n+1)} - \tau^{(n)}), T_{i+1}] - x[\tau^{(n)} + \theta(\tau^{(n+1)} - \tau^{(n)}), T_i]}$$

At $\theta = 0$ we obtain an explicit approximation of the initial differential operator:

$$\int_{\tau^{(n)}}^{\tau^{(n+1)}}\left\{\int_{T_{i-1/2}}^{T_{i+1/2}}\frac{\partial}{\partial T}\left[-\frac{\lambda(T)}{\partial x/\partial T}\right]\mathrm{d}T\right\}\mathrm{d}\tau$$

$$\approx \frac{\lambda_{i-1/2}(T_i-T_{i-1})\Delta\tau}{x_i^{(n)}-x_{i-1}^{(n)}}-\frac{\lambda_{i+1/2}(T_{i+1}-T_i)\Delta\tau}{x_{i+1}^{(n)}-x_i^{(n)}},$$

and at $\theta = 1$ its implicit approximation:

$$\int_{\tau^{(n)}}^{\tau^{(n+1)}}\left\{\int_{T_{i-1/2}}^{T_{i+1/2}}\frac{\partial}{\partial T}\left[-\frac{\lambda(T)}{\partial x/\partial T}\right]\mathrm{d}T\right\}\mathrm{d}\tau$$

$$\approx \frac{\lambda_{i-1/2}(T_i-T_{i-1})\Delta\tau}{x_i^{(n+1)}-x_{i-1}^{(n+1)}}-\frac{\lambda_{i+1/2}(T_{i+1}-T_i)\Delta\tau}{x_{i+1}^{(n+1)}-x_i^{(n+1)}}.$$

The last integral from equation (57) is calculated in the following way:

$$\int_{T_{i-1/2}}^{T_{i+1/2}}\left[\int_{\tau^{(n)}}^{\tau^{(n+1)}}\frac{\lambda(T)}{l_0-x}\mathrm{d}\tau\right]\mathrm{d}T$$

$$=\int_{T_{i-1/2}}^{T_{i+1/2}}\left[\lambda(T)\int_{\tau^{(n)}}^{\tau^{(n+1)}}\frac{\mathrm{d}\tau}{l_0-x}\right]\mathrm{d}T,$$

then with the approximation by the explicit scheme we have:

$$\int_{\tau^{(n)}}^{\tau^{(n+1)}}\frac{\mathrm{d}\tau}{l_0-x}\approx\frac{\Delta\tau}{l_0 x^{(n)}(T)}$$

and with the approximation by the implicit scheme:

$$\int_{\tau^{(n)}}^{\tau^{(n+1)}}\frac{\mathrm{d}\tau}{l_0-x}\approx\frac{\Delta\tau}{l_0-x^{(n+1)}(T)}.$$

Then we obtain, correspondingly, with the explicit approximation scheme:

$$\int_{T_{i-1/2}}^{T_{i+1/2}}\lambda(T)\frac{\Delta\tau}{l_0-x^{(n)}}\mathrm{d}T=\Delta\tau\int_{T_{i-1/2}}^{T_{i+1/2}}\frac{\lambda(T)\,\mathrm{d}T}{l_0-x^{(n)}}$$

$$\approx\frac{\Delta\tau\lambda_i}{l_0-x_i^{(n)}}(T_{i+1/2}-T_{i-1/2})$$

$$\lambda_i=\int_{T_{i-1/2}}^{T_{i+1/2}}\lambda(T)\,\mathrm{d}T/(T_{i+1/2}-T_{i-1/2}).$$

If the last integral is replaced by $\lambda(T_i)(T_{i+1/2}-T_{i-1/2})$, then we have $\lambda_i=\lambda(T_i)$. As a result we obtain with the explicit approximation:

$$\int_{T_{i-1/2}}^{T_{i+1/2}}\left[\lambda(T)\int_{\tau^{(n)}}^{\tau^{(n+1)}}\mathrm{d}\tau/(l_0-x)\right]\mathrm{d}T$$

$$\approx\lambda_i\Delta\tau(T_{i+1/2}-T_{i-1/2})/(l_0-x^{(n)})$$

and with the implicit approximation:

$$\int_{T_{i-1/2}}^{T_{i+1/2}}\left[\lambda(T)\int_{\tau^{(n)}}^{\tau^{(n+1)}}\mathrm{d}\tau/(l_0-x)\right]\mathrm{d}T$$

$$\approx\lambda_i\Delta\tau(T_{i+1/2}-T_{i-1/2})/(l_0-x^{(n+1)}).\quad(58)$$

Substitution of all of the results obtained in equation (57) after elementary transformations by the explicit approximation scheme yields

$$C_i\frac{x_i^{(n+1)}-x_i^{(n)}}{\Delta\tau}=\frac{2\lambda_{i-1/2}(T_i-T_{i-1})}{(x_i^{(n)}-x_{i-1}^{(n)})(T_{i+1}-T_{i-1})}-\frac{2\lambda_{i+1/2}(T_{i+1}-T_i)}{(x_{i+1}^{(n)}-x_i^{(n)})(T_{i+1}-T_{i-1})}+\frac{(m-1)\lambda_i}{l_0-x_i^{(n)}},\quad(59)$$

and approximation by the implicit scheme gives

$$C_i\frac{x_i^{(n+1)}-x_i^{(n)}}{\Delta\tau}=\frac{2\lambda_{i-1/2}(T_i-T_{i-1})}{(x_i^{(n+1)}-x_{i-1}^{(n+1)})(T_{i+1}-T_{i-1})}-\frac{2\lambda_{i+1/2}(T_{i+1}-T_i)}{(x_{i+1}^{(n+1)}-x_i^{(n+1)})(T_{i+1}-T_{i-1})}+\frac{(m-1)\lambda_i}{l_0-x_i^{(n+1)}}.\quad(60)$$

5.2. *Analysis of the stability of digitization schemes*

Computer solution of the finite difference analogue of equation (56′) with its approximation by explicit or implicit schemes is possible when stability of calculations is provided. This requires the establishment of such a ratio between the time steps $\Delta\tau$ and temperature ΔT at which the error of the method remains small during calculations. Determination of the condition of the stability for the equation under consideration by familiar methods [12] is very difficult due to its non-linearity. It can be shown that the analysis of the solution scheme stability may also be performed in this case.

In fact, consider equation (56′) at $m = 1$, $C(T) =$ const., $\lambda(T) =$ const. and replace x by u as in the majority of works on this problem.

Then we have

$$\frac{C}{\lambda}\frac{\partial u(\tau,T)}{\partial T}=-\frac{\partial}{\partial T}\left(\frac{1}{u'_T}\right)=(u'_T)^{-2}\frac{\partial^2 u}{\partial T^2}.\quad(61)$$

'Freezing' the multiplier $(u'_T)^{-2}$ on the right-hand side of equation (61) and denoting it in terms of D, we obtain

$$\frac{C}{\lambda}\frac{\partial u(\tau,T)}{\partial\tau}=D\frac{\partial^2 u}{\partial T^2}.\quad(62)$$

As is known from ref. [12], the condition of explicit scheme stability for equation (62) is

$$\frac{\lambda}{C}\frac{\Delta\tau}{(\Delta T)^2}\leqslant\tfrac{1}{2}D\quad\text{or}\quad\frac{\lambda}{C}\frac{\Delta\tau}{(\Delta T)^2}\leqslant\tfrac{1}{2}(u'_T)^{-2}.\quad(63)$$

The finite difference analogue of the right-hand side of equation (63) may be obtained by transforming equation (61) in the following way:

$$\frac{C}{\lambda}\frac{u_i^{(n+1)}-u_i^{(n)}}{\Delta\tau^{(n)}}=-\left(\frac{T_{i+1}-T_i}{u_{i+1}^{(n)}-u_i^{(n)}}-\frac{T_{i-1}T_i}{u_i^{(n)}-u_{i-1}^{(n)}}\right)\frac{2}{T_{i+1}-T_{i-1}}$$

$$= \left[\frac{(u_{i+1}^{(n)}-u_i^{(n)})(T_i-T_{i-1})-(u_i^{(n)}-u_{i-1}^{(n)})(T_{i+1}-T_i)}{(u_{i+1}^{(n)}-u_i^{(n)})(u_i^{(n)}-u_{i-1}^{(n)})}\right] \times \frac{2}{T_{i+1}-T_{i-1}}$$

$$= \left(\frac{u_{i+1}^{(n)}-u_i^{(n)}}{T_{i+1}-T_i}-\frac{u_i^{(n)}-u_{i-1}^{(n)}}{T_i-T_{i-1}}\right)\frac{1}{(T_{i+1}-T_{i-1})/2} \times\left(\frac{u_{i+1}^{(n)}-u_i^{(n)}}{T_{i+1}-T_i}\times\frac{u_i^{(n)}-u_{i-1}^{(n)}}{T_i-T_{i-1}}\right)^{-1}. \quad (64)$$

In equation (64) the co-factors

$$\left(\frac{u_{i+1}^{(n)}-u_i^{(n)}}{T_{i+1}-T_i}-\frac{u_i^{(n)}-u_{i-1}^{(n)}}{T_i-T_{i-1}}\right)\frac{1}{(T_{i+1}-T_{i-1})/2},$$

$$\left(\frac{u_{i+1}^{(n)}-u_i^{(n)}}{T_{i+1}-T_i}\times\frac{u_i^{(n)}-u_{i-1}^{(n)}}{T_i-T_{i-1}}\right)^{-1}$$

are the finite difference analogues of $\partial^2 u/\partial T^2$ and $(u'_T)^{-2} = D$ in equation (61).

Then, based on equation (63) and the considerations made, we obtain the following condition of explicit scheme stability for equation (62) and, consequently, equation (61):

$$\frac{\lambda}{C}\Delta\tau^{(n)} \leqslant \frac{1}{2}\min_i (T_{i+1}-T_i)^2 \times \min_i\left(\frac{u_{i+1}^{(n)}-u_i^{(n)}}{T_{i+1}-T_i}\,\frac{u_i^{(n)}-u_{i-1}^{(n)}}{T_i-T_{i-1}}\right). \quad (65)$$

With the temperature-dependent thermophysical properties, condition (65) takes the form

$$\Delta\tau^{(n)} \leqslant \frac{1}{2}\frac{C_{\min}}{\lambda_{\max}}\min_i (T_{i+1}-T_i)^2 \times \min_i\left(\frac{u_{i+1}^{(n)}-u_i^{(n)}}{T_{i+1}-T_i}\,\frac{u_i^{(n)}-u_{i-1}^{(n)}}{T_i-T_{i-1}}\right) \quad (66)$$

where $C_{\min}$ and $\lambda_{\max}$ are the minimum bulk heat capacity and the maximum thermal conductivity within the range of temperatures on the nth time step.

It is easy to see that condition (66) imposed on the time is very 'rigorous' though it guarantees stability of calculations by the explicit scheme.

A less burdensome condition imposed on the quantity $\Delta\tau^{(n)}$ may be obtained by applying the maximum principle to equation (59), written in the form

$$\frac{C}{\lambda}\frac{u_i^{(n+1)}-u_i^{(n)}}{\Delta\tau^{(n)}} = a_i^{(n)}\left(\frac{u_{i+1}^{(n)}-u_i^{(n)}}{T_{i+1}-T_i} - \frac{u_i^{(n)}-u_{i-1}^{(n)}}{T_i-T_{i-1}}\right)\frac{2}{T_{i+1}-T_{i-1}},$$

where

$$a_i^{(n)} = \frac{T_{i+1}-T_i}{u_{i+1}^{(n)}-u_i^{(n)}}\,\frac{T_i-T_{i-1}}{u_i^{(n)}-u_{i-1}^{(n)}}.$$

Let $\delta u_i^{(n)}$ be some variation of u_i on the nth time layer. By 'freezing' $a_i^{(n)}$, we obtain the following equation for $\delta u_i^{(n+1)}$:

$$\frac{C}{\lambda}\frac{\delta u_i^{(n+1)}-\delta_i^{(n)}}{\Delta\tau^{(n)}} = a_i^{(n)}\left(\frac{\delta u_{i+1}^{(n)}-\delta u_i^{(n)}}{T_{i+1}-T_i} - \frac{\delta u_i^{(n)}-\delta u_{i-1}^{(n)}}{T_i-T_{i-1}}\right)\frac{2}{T_{i+1}-T_{i-1}}$$

and

$$\delta u_i^{(n)} = \frac{2a_i^{(n)}\Delta\tau^{(n)}\lambda\delta u_{i+1}^{(n)}}{(T_{i+1}-T_i)(T_{i+1}-T_{i-1})C} + \left[1-\frac{2a_i^{(n)}}{(T_{i+1}-T_i)(T_{i+1}-T_{i-1})} - \frac{2a_i^{(n)}}{(T_i-T_{i-1})(T_{i+1}-T_{i-1})}\right]\frac{\Delta\tau^{(n)}\lambda}{C}\delta u_i^{(n)} + \frac{2a_i^{(n)}\Delta\tau^{(n)}\lambda\delta u_{i-1}^{(n)}}{(T_{i+1}-T_{i-1})(T_i-T_{i-1})}$$

or

$$\delta u_{i+1}^{(n)} = \beta_1\delta u_{i+1}^{(n)} + \beta_2\delta u_i^{(n)} + \beta_3\delta u_{i-1}^{(n)}. \quad (67)$$

In equation (67), $\beta_1 > 0$, $\beta_3 > 0$ and $\beta_1+\beta_3+\beta_2 = 1$. To the stability condition of the solution of equation (67), $\beta_2 \geqslant 0$, there is equivalently the fulfilment of the inequality

$$\frac{\lambda}{C}\frac{a_i^{(n)}\Delta\tau^{(n)}}{(T_{i+1}-T_{i-1})}\left(\frac{1}{T_i-T_{i-1}}+\frac{1}{T_{i+1}-T_i}\right) \leqslant \tfrac{1}{2}$$

or

$$\frac{\lambda}{C}\Delta\tau^{(n)}\max_i\left[\frac{a_i^{(n)}}{T_{i+1}-T_{i-1}}\left(\frac{1}{T_{i+1}-T_i}+\frac{1}{T_{i-1}-T_i}\right)\right] \leqslant \tfrac{1}{2}. \quad (68)$$

With temperature-dependent λ and C, condition (68) takes the form

$$\frac{\lambda_{\max}}{C_{\min}}\Delta\tau^{(n)}\max_i\left[\frac{a_i^{(n)}}{T_{i+1}-T_{i-1}}\left(\frac{1}{T_{i+1}-T_i}+\frac{1}{T_{i-1}-T_i}\right)\right] \leqslant \tfrac{1}{2}.$$

Similarly, by applying the maximum principle to the analysis of the implicit scheme (60) we obtain its absolute stability at any $\Delta\tau$ to ΔT ratio.

5.3. *Computer realization of the method*

The determination of the $\Delta\tau$ to ΔT ratio at each time step, which provides the stability of the solution by the explicit scheme (59), could cover the problems of its computer realization if the right-hand side of equation (59) were divergent. It is non-divergent due to the presence of (56) on the right-hand side of (59) of the sum $(m-1)\lambda(T)/T$, thus leading to inaccurate results [11].

The computer solution of the system of equations by the implicit scheme does not show the principle restrictions imposed on the relationship between the steps $\Delta\tau$ and ΔT but there arise difficulties due to the non-linear character of (60).

With computer calculation the first of the above difficulties is overcome by the application of the split of equation (56), when on each time layer the equations

$$C(T)\frac{\partial x}{\partial \tau} = \frac{\partial}{\partial T}\left[-\frac{\lambda(T)}{\partial x/\partial T}\right] \qquad (69)$$

$$C(T)\frac{\partial x}{\partial \tau} = \frac{(m-1)\lambda(T)}{l_0 - x} \qquad (70)$$

are solved successively using finite differences. The latter equation may be rewritten in divergent form:

$$C(T)\frac{\partial(l_0-x)^2}{\partial \tau} = -2(m-1)\lambda(T). \qquad (71)$$

In the differential form with the value of the time half-layer $\Delta\tau$, equations (69) and (71) become

$$C_i\frac{x_i^{(n+1/2)}-x_i^{(n)}}{\Delta\tau} = \frac{\Delta}{\Delta T}\left(\frac{\lambda_i}{\Delta x^{(n+1/2)}/\Delta T}\right) \qquad (72)$$

$$C_i\frac{(l_0-x_i^{(n+1)})^2-(l_0-x_i^{(n+1/2)})^2}{\Delta\tau} = -2\lambda_i(m-1). \qquad (73)$$

The operator on the right-hand side of equation (72) is given in Section 5.1.

The sequence of computer calculations on each time layer is the following: solve the system of equations concerned with digitization of equation (72), find $x_i^{(n+1/2)}$ $(i = N,\ N-1, \ldots, 1)$ and then determine $x_i^{(n+1)}$ explicitly from equation (73) using the formula

$$x_i^{(n+1)} = l_0 - [(l_0 - x_i^{(n+1/2)})^2 - 2\lambda_i(m-1)\Delta\tau/C_i]^{1/2}. \qquad (74)$$

Here, the minus sign in front of the square root is due to the fact that $x_i^{(n+1/2)} < l_0$ and for $\Delta\tau \to 0$ we should have $x_i^{(n+1)} = x_i^{(n+1/2)}$. The system of non-linear equations (72), expressed in the form

$$(x_i^{(n+1/2)}-x_i^{(n)})(x_i^{(n+1/2)}-x_{i+1}^{(n+1/2)})(x_{i-1}^{(n+1/2)}-x_i^{(n+1/2)}) - d_i(x_{i-1}^{(n+1/2)}-x_i^{(n+1/2)}) + b_i(x_i^{(n+1/2)}-x_{i+1}^{(n+1/2)}) = 0, \qquad (75)$$

where

$$d_i = \frac{2\Delta\tau\lambda_{i+1/2}(T_{i+1}-T_i)}{C_i(T_{i+1}-T_{i-1})},$$

$$b_i = \frac{2\Delta\tau\lambda_{i-1/2}(T_i-T_{i-1})}{C_i(T_{i+1}-T_{i-1})},$$

is solved by the Newton–Raphson iterative method, characterized by rapid convergence of approximations [12].

In this case it is closed by a difference analogue of the boundary conditions,

$$\lambda_{N+1/2}\frac{T_N-T_{N+1}}{x_N^{(n+1)}} + \alpha^{(n+1)}(T_{N+1}-T_f^{(n+1)}) = q^{(n+1)}, \qquad (76)$$

in which with the assignment of boundary conditions of the first kind, the temperature T_{N+1} of the body bounding surface with the coordinate $x_{N+1} = 0$ (Fig. 2) coincides with the prescribed function $T_f^{(n+1)}$, since the coefficient of convective heat transfer is assumed to be infinitely large $(\alpha^{(n+1)} \to \infty)$; with boundary conditions of the second kind, we have $\alpha^{(n+1)} = 0$ and the density of the heat flux $q^{(n+1)}$ into the body bounding surface is known; in the case of boundary conditions of the third kind, $\alpha^{(n+1)}$ and the surrounding medium temperature $T_f^{(n+1)}$ are known and $q^{(n+1)}$ is assumed to be equal to zero.

The initial distribution of the unknown quantity $x_i^{(0)}$ should be known. In the case of uniform initial temperature distribution it should be prescribed artificially as non-uniform in a thin layer adjacent to the body bounding surface, for example, by a quadratic parabola. When solving a two-phase Stefan problem on body melting or solidification the initial temperature distribution in the parent phase should be known and if at the start of calculation there is already a melted or solidified layer then the initial distribution should be prescribed in it.

By using the above algorithm, programs for calculating temperature fields in bodies with and without phase transition of substance were created. Figure 3 gives a schematic diagram of the calculation program using the implicit scheme of digitization of differential operators in equation (56′). In the modulus of the program, bifurcations were allowed by the criterion of the presence or absence of phase transition and within each bifurcation calculations on a half-line or on a segment were separated.

The program was initially tested when solving a number of model problems of non-stationary heat conduction in a plate, cylinder and sphere and in a half-space without phase transition of the substance with boundary conditions of first, second and third order.

In this case computational data with the shift of isothermal surfaces fully corresponded to the data obtained by the grid method in the traditional treatment of the process as spatial–temporal temperature variation. On a half-line the data of numerical calculations at $C = 1$, $\lambda = (1-0.8T)^{-2}$, $\lambda = (1-3.2927T+2.877T^2)^{-2}$ completely coincided with Fujita's accurate solutions [10].

For a case with substance phase transition the known model problem of melting in a half-space was solved on a computer. The initial temperature of the solid phase was taken to be $T_{2,i}^{(0)} = -1$, the phase transition temperature was $-T_s = 0$ and the bounding surface temperature was assumed as $T_{N+1} = 1$.

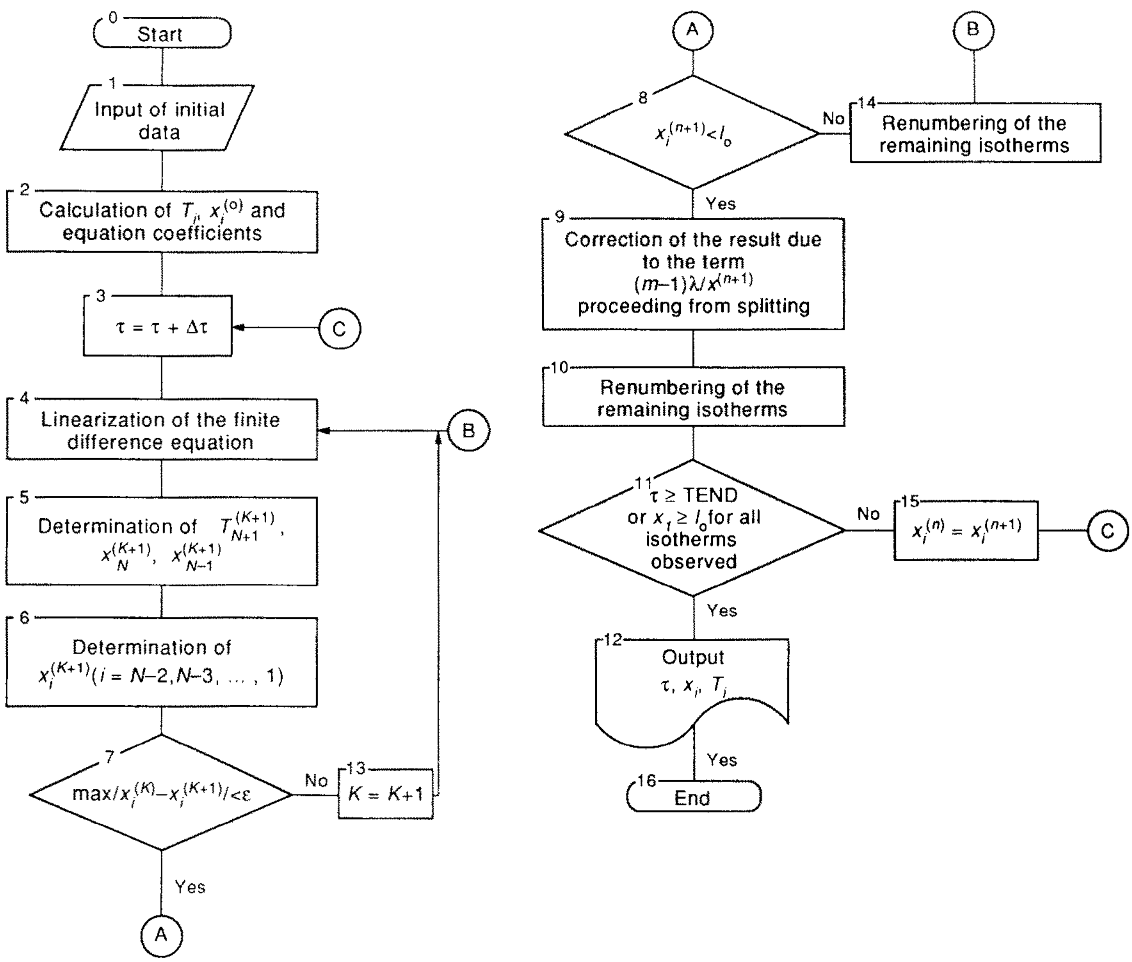

FIG. 3. Schematic diagram of the program of thermal calculation in the migration of isotherms.

Thermophysical properties in the material solid phase are $C_2 = 1$, $\lambda_2 = 5$ and in the liquid phase $C_1 = 20$, $\lambda_1 = 1$. Moreover, the ratio of bulk heat phase transition, L, to specific heat of a solid phase was taken as $L/C_2 = 0.2$.

The coordinates $x_s^{(n+1)}$ of the phase interface and the dimensionless coordinates $x_{1,i}^{(n+1)}/x_s^{(n+1)}$ for temperatures 0.1, 0.2, ..., 1.0 in the new phase and $x_{2,i}^{(n+1)}/x_s^{(n+1)}$ for temperatures -1.0, -0.9, ..., -0.1 in the old phase were printed.

A good agreement between the results calculated by the implicit scheme of the described two-phase Stefan problem and the data of the accurate solution [13] (Fig. 4) indicates the efficiency of the developed method (the difference between the values of x_s, $x_{1,i}/x_s$, $x_{2,i}/x_s$ and accurate ones did not exceed 3% though this cannot be explained by the fact that in our algorithm free convection in a liquid phase was not taken into account).

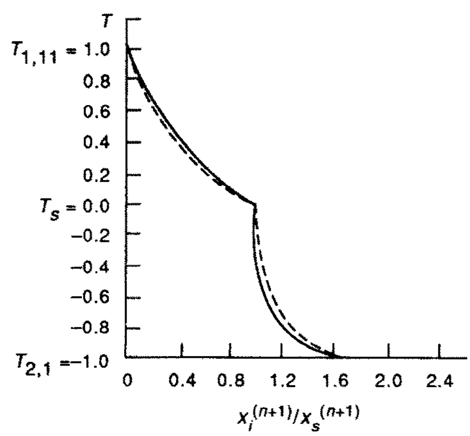

FIG. 4. Exact (solid line) and numerical (dashed line) solutions of the model Stefan problem for boundary conditions of the first kind.

Then the model Stefan problem of half-space solidification at $\lambda = C = L = 1$, when the initial temperature field in a new phase is described by a distribution of the type $T_0^{(1)} = \exp(-x) - \exp(-b)$ (b is the new phase thickness at $\tau = 0$) and the temperature of substance phase transition and power of volumetric heat release sources are equal to zero, was solved on a computer using the developed method.

In this case it is assumed that solidification occurs with fixed heat removal from the outer surface of a new phase into the surrounding medium (boundary conditions of the second kind) and from the zone of phase transition into the old phase, respectively, according to the formulae

$$q|_{x=0} = \exp \tau, \quad q|_{x_s+0} = \exp(-b) - 1.$$

The data of numerical calculation coincided with the known exact solution for temperature distribution and location of the phase interface, which have the form

$$T^{(1)} = \exp(-x+\tau) - \exp(-b), \quad x_s = b+\tau.$$

It should be noted that the duration of numerical calculations of the Stefan-type problems by the developed technique using the implicit scheme turned out to be two orders smaller than those in the familiar methods of phase interface 'catching' into the grid node, 'straightening' of boundaries, etc. [14–21]. This allows one to make a great number of variant solutions of casting solidification. The latter point is also important in numerical study of the processes of substance phase transition of great duration in a real time scale.

6. CONCLUSION

Consideration of the process of non-stationary heat conduction in the migration of isothermal surfaces made it possible not only to determine its new regularities and specific features but also to develop effective techniques for numerical–analytical determination of temperature fields in solid bodies as well as the location of the boundary between new and old phases during their melting or solidification.

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METHODE DE LA MIGRATION DES ISOTHERMES DANS L'ETUDE DU TRANSFERT DE CHALEUR ET DE MASSE EN THEORIE ET EN PRATIQUE—II. DETERMINATION NUMERIQUE-ANALYTIQUE DES CHAMPS DE TEMPERATURE

Résumé—L'équation de la conduction thermique variable dans les corps solides, une nouvelle équation de la physique mathématique, est trouvée pour un cas multidimensionnel de migration des surfaces isothermes. On donne l'analyse de l'existence et de l'unicité de la solution du problème. L'application de la méthode des perturbations, dans la description cinématique de la conduction thermique, à la solution du problème est faite avec des propriétés thermophysiques dépendant de la température. L'approximation par différence finie de l'équation nouvelle est trouvée et on justifie la condition de stabilité. En utilisant ces schémas, on développe un algorithme pour déterminer par ordinateur les champs de température et la localisation de l'interface entre nouvelle et vieille phases.

DAS VERFAHREN DER ISOTHERMENWANDERUNG IN THEORIE UND PRAXIS DER WÄRME- UND STOFFTRANSPORTUNTERSUCHUNG—II. NUMERISCH-ANALYTISCHE BESTIMMUNG VON TEMPERATURFELDERN

Zusammenfassung—Die grundlegende Gleichung für den Vorgang der nichtstationären Wärmeleitung in Festkörpern—eine neue Gleichung der mathematischen Physik—wird für einen mehrdimensionalen Fall der Wanderung isothermer Oberflächen gefunden. Die Existenz und die Eindeutigkeit der Lösung des ersten Randwertproblems für die obige Gleichung wird gezeigt. Das Verfahren der Störungen der kinematischen Beschreibung der Wärmeleitung wird bei der Lösung des Problems mit temperaturbhängigen thermophysikalischen Stoffeigenschaften angewandt. Es wird die Finite-Differenzen-Näherung der neuen Wärmeleitgleichung durch Randwerte und explizite Formulierung ermittelt und eine Stabilitätsbedingung für letztere hergeleitet. Unter Verwendung dieser Formulierung wird der Algorithmus abgeleitet, um Temperaturfelder und den Ort der Grenzfläche zwischen neuen und alten Phasen mit Hilfe eines Computers ermitteln zu können.

МЕТОД ПЕРЕМЕЩЕНИЯ ИЗОТЕРМ В ТЕОРИИ И ПРАКТИКЕ ТЕПЛОМАССОПЕРЕНОСА. II. ЧИСЛЕННО-АНАЛИТИЧЕСКОЕ ОПРЕДЕЛЕНИЕ ТЕМПЕРАТУРНЫХ ПОЛЕЙ

Аннотация—Установлено уравнение процесса нестационарной теплопроводности в твердых телах для многомерного случая в перемещениях изотермических поверхностей—новое уравнение математической физики и дан анализ существования и единственности решения первой краевой задачи для него. Показано использование метода возмущений при кинематическом описании процесса теплопроводности для получения решения задач с зависящими от температуры теплофизическими свойствами среды. Установлена конечно-разностная аппроксимация нового уравнения теплопроводности по краевой и явной схемам и обосновано условие устойчивости последней. С их использованием разработан алгоритм определения на ЭВМ температурных полей и местоположения границы раздела новой и старой фаз.